Last time: $G \curvearrowright X$, $g \cdot x = \Phi_g(x)$

group set

orbit (of $x \in X$): $G \cdot x = \{ g \cdot x \in X \mid g \in G \}$

stabilizer (of $x \in X$): $\mathrm{Stab}_G(x) = \{ g \in G \mid g \cdot x = x \}$

O-S theorem: $|G \cdot x| = \dfrac{|G|}{|\mathrm{Stab}_G(x)|}$

adjoint action $G \curvearrowright G$
- orbits are conjugacy class
- stabilizers are centralizer

Class equation (of G): $|G| = \sum_{\substack{\text{conj.} \\ \text{classes } C}} |C| = \sum_{\substack{\text{conj} \\ \text{classes } C}} \dfrac{|G|}{|C_G(g)|}$, $C \ni g$

This class: $H \leq G \rightsquigarrow$ left cosets gH

right cosets Hg ?

<u>Def</u>: H is called a normal subgroup of G ($H \trianglelefteq G$)

if $gH = Hg$, $\forall g \in G$

$$\Updownarrow$$

$$gHg^{-1} = H \quad \text{(preserved under conjugation)}$$

<u>Prop</u>: $\forall$ group homomorphism $f: G \longrightarrow G'$,

$$K := \operatorname{Ker} f \trianglelefteq G$$

<u>Proof</u>: need to check $gKg^{-1} = K$, $\forall g \in G$

$$\Updownarrow$$

$(gKg^{-1} = \{gkg^{-1} \mid k \in K\})$

$f(gKg^{-1}) = \{f(gkg^{-1}) \mid k \in K\}$

$$f(gKg^{-1}) = e' \in G', \quad \forall g \in G$$

$$\Updownarrow$$

$$\underbrace{f(g)}_{f(g)} \underbrace{f(K)}_{e'} \underbrace{f(g^{-1})}_{f(g)^{-1}} = e', \quad \forall g \in G \quad \checkmark$$

Quotient groups: $H \trianglelefteq G$,

$G/_H = {}_H\backslash G$ inherits a group structure from G

$\trianglelefteq$

$[g] \cdot [g'] = [gg']$, note $[g] = [gh] = [hg]$ $\forall g \in G, \forall h \in H$

$$\begin{array}{ccccc} [g] & \cdot & [g'] & = & [gg'] \\ \| & & \| & & \| \text{ H is normal} \\ [gh] & \cdot & [g'h'] & = & [ghg'h'] \end{array}$$

Basic properties:

- $H_1, H_2 \trianglelefteq G \Longrightarrow H_1 \cap H_2 \trianglelefteq G$
- $H \trianglelefteq G \Longrightarrow H \trianglelefteq$ (any subgroup of G which contains H)

• $H \trianglelefteq G$, $\pi : G \longrightarrow G/H$, $g \longrightarrow [g]$

is a group homomorphism with kernel H

$$\pi(g) = \text{identity}$$
$$[g] = [e] \iff g \in H$$

• First isomorphism theorem:

if $f : G \longrightarrow G'$ is a group homomorphism,

then f induces an isomorphism:

$$G/\mathrm{Ker}\, f \cong \mathrm{Im}\, f$$

$G/\mathrm{Ker}\, f$: a quotient of G ; $\mathrm{Im}\, f$: a subgroup of G'

$$[g] \rightsquigarrow f(g)$$
$$\| \qquad\qquad \|$$
$$[gh] \rightsquigarrow f(gh) = f(g)\underbrace{f(h)}_{e'}, \ \forall h \in \mathrm{Ker}\, f$$

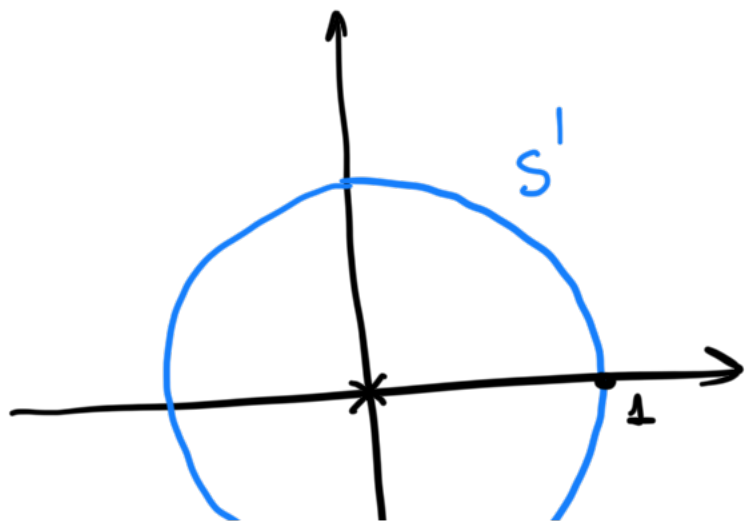
Ex: $G = (\mathbb{R}, +)$
$G' = (\mathbb{C}^*, \times)$

$f : \mathbb{R} \longrightarrow \mathbb{C}^* \ni 1$

$f(x) = e^{2\pi i x} = \cos(x) + i \cdot \sin(x)$

$\mathrm{Ker}\, f = \{x \in \mathbb{R} \mid e^{2\pi i x} = 1\} = \mathbb{Z}$

$\mathrm{Im}\, f = \text{unit circle} = S^1$



1^{st} iso th : $\boxed{\mathbb{R}/\ \cong S^1}$

1 iso theorem says $\boxed{\mathbb{R}/\mathbb{Z} = S}$

Let G be any group and let $X \subseteq G$ any subset

$$C_G(X) = \{ g \in G \mid g x g^{-1} = x, \ \forall x \in X \}$$

centraliser

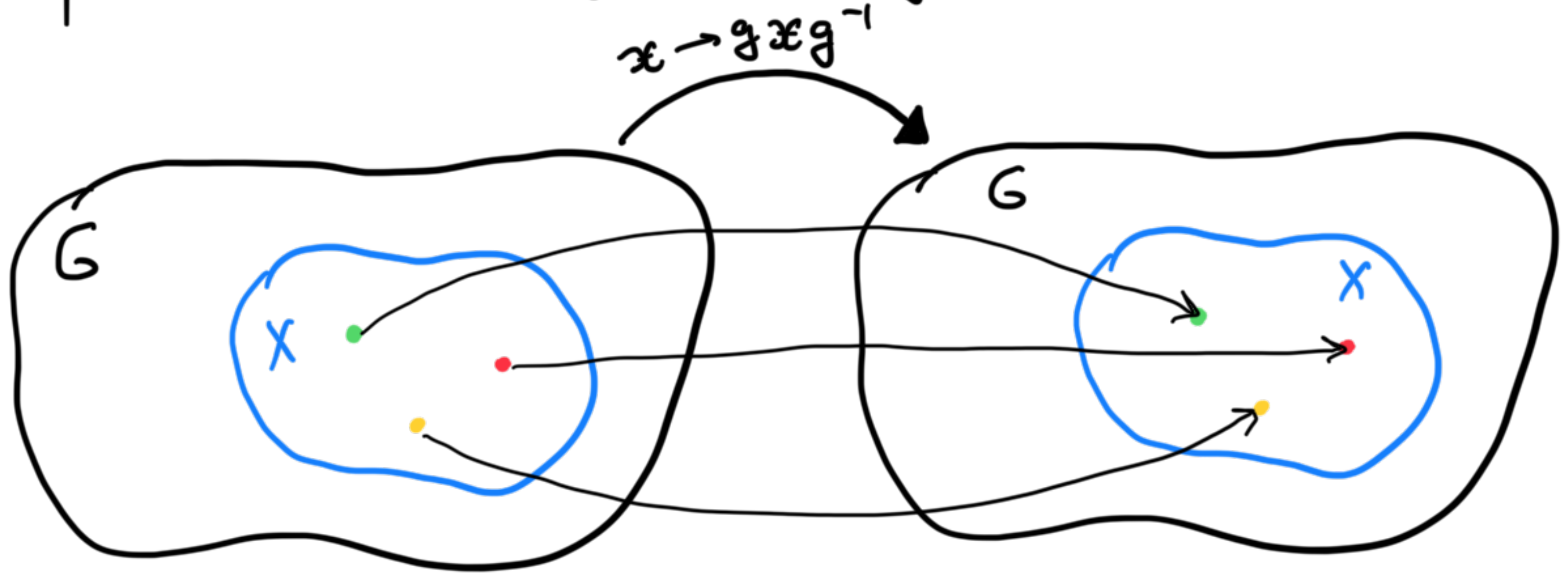
$$\cap \text{I}$$

$$N_G(X) = \{ g \in G \mid g X g^{-1} = X \}$$

normaliser

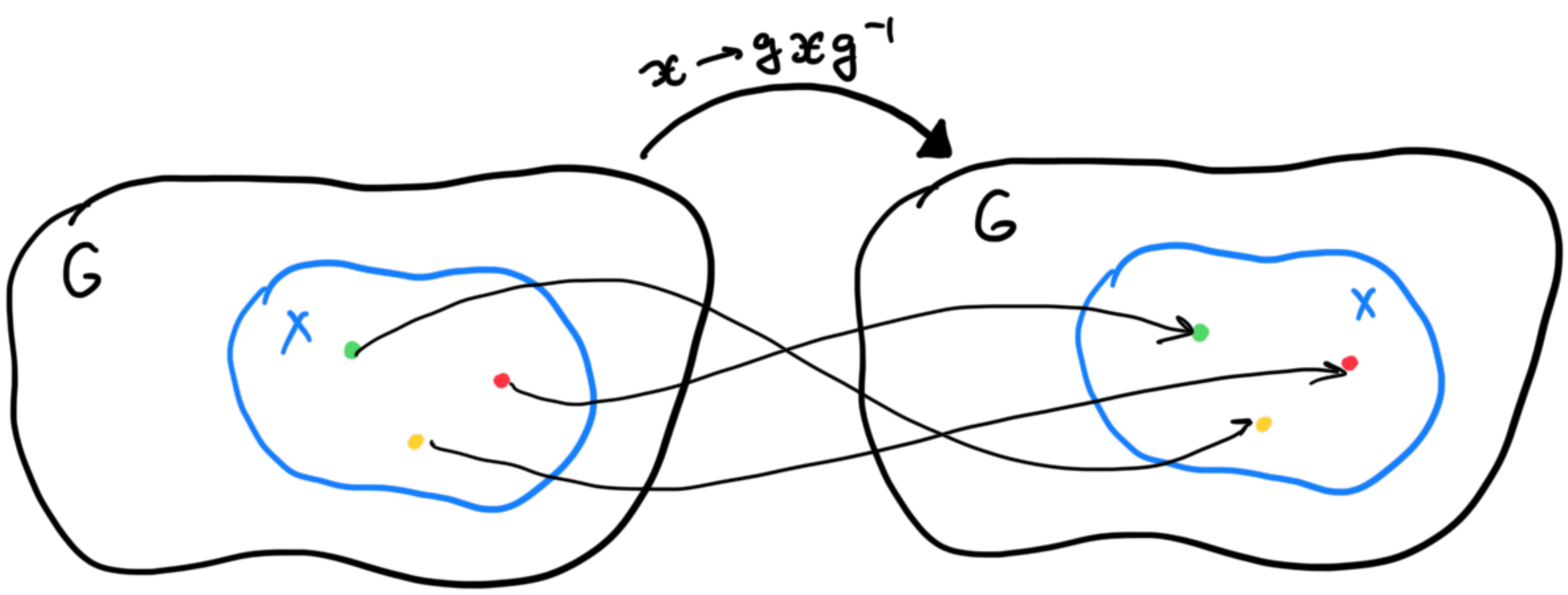
Explanation using the adjoint action $G \curvearrowright G$; take $g \in G$

if



then $g \in C_G(X)$

if



then $g \in N_G(X)$

Prop: $C_G(X) \trianglelefteq N_G(X)$

Proof: using actions

Interlude: $G \curvearrowright X$ any action on any set;

$$g \cdot x = \Phi_g(x)$$

$$H = \{h \in G \mid \Phi_h = Id_X\} = \text{Kernel of action}$$

must be normal $\trianglelefteq G$

$$= \operatorname{Ker}(G \longrightarrow S_X)$$

<u>Claim:</u> induced action $G/_H \curvearrowright X$

faithful

$$[g]\cdot x = g \cdot x \qquad \forall h \in H$$

$$\begin{matrix} \| & & \| \\ [gh]\cdot x & = (gh)\cdot x = & g \cdot (\underbrace{h \cdot x}_{x}) \end{matrix}$$

Return to proof of Proposition

$$\begin{matrix} G \curvearrowright & G \\ & \cup\!\vert \\ & X \end{matrix}$$

adjoint action might not send X to X

$N_G(X) \curvearrowright G$ adjoint action sends X to X by definition

$$\Downarrow$$

$N_G(X) \curvearrowright X$; this action has kernel $C_G(X)$

$$\Downarrow$$

$$C_G(X) \trianglelefteq N_G(X)$$

Also, we get an induced action

$$N_G(X)/C_G(X) \curvearrowright X$$

$G / C_G(X)$

What if X is a subgroup of G to begin with?
Then this action will be by automorphism of X

<u>Def</u>: $\forall$ group K, an isomorphism $f: K \longrightarrow K$ is called an automorphism

<u>Prop</u>: $Aut(K) = \{$ automorphisms $f: K \longrightarrow K \}$ is a group w.r.t. (with respect to) composition

relies on the fact that composition / inverses of autos are autos

Ex: $f: G \longrightarrow G$, $f(g) = g^{-1}$ is this an automorphism? $\Leftrightarrow$ G is abelian

$f \circ f = Id_G$

$$h^{-1} g^{-1} = (gh)^{-1} = f(gh) = f(g) f(h) = g^{-1} h^{-1}$$

Def: $G \curvearrowright K$ is an action by automorphisms if every $\Phi_g : K \longrightarrow K$ is an automorphism, $\forall g \in G$

groups   groups

i.e. $g \cdot (kk') = (g \cdot k)(g \cdot k')$

Ex: $\mathbb{Z}/2\mathbb{Z} \curvearrowright K$ by automorphisms

involution, i.e. $f \circ f = Id_K$

$0 \bmod 2 \rightsquigarrow \Phi_0 = Id_K$

$1 \bmod 2 \rightsquigarrow \Phi_1$ needs to be an automorphism $f: K \to K$

$1+1=0 \rightsquigarrow \Phi_1 \circ \Phi_1 = \Phi_0 = Id_K$

(e.g. $\mathbb{Z}/2\mathbb{Z} \curvearrowright$ abelian group G by having f be the inverse function)

Thm: (Normaliser / Centraliser)

if $H \leq G$, then the action

$$N_G(H)/C_G(H) \curvearrowright H$$

is an action by automorphisms

$\Updownarrow$

$$N_G(H)/C_G(H) \xhookrightarrow{\Phi} Aut(H)$$ is an injective group homomorphism

Proof: must show that red action is by automorphism of H, i.e.

$[g] \in N_G(H)/C_G(H)$ represented by $g \in N_G(H)$

$[g] \cdot h = g h g^{-1}$ (inverse assignment is $[g^{-1}] \cdot h = g^{-1} h g$)

$$[g] \cdot hh' \overset{\text{want}}{=} ([g] \cdot h)([g] \cdot h')$$

$$\| \qquad\qquad\qquad \|$$

$$ghh'g^{-1} = gh\underbrace{g^{-1}g}_{e}h'g^{-1}$$

Reminder: $H, K \leq G$

Remember: $H, K \leq G$

$HK = \{hk \mid h \in H, k \in K\}$
$KH = \{kh \mid h \in H, k \in K\}$ are subsets of G

Facts • $HK = KH$ iff HK is a subgroup of G

(quick proof: $\underbrace{HH}_{H}\underbrace{KK}_{K} = HKHK \iff HK$ is a subgroup)

• $H, K \trianglelefteq G$, then $HK = KH$ is a normal subgroup of G
(b\c $Hk = kH, \forall k \in K$)

(a scenario in which $HK = KH$ holds is when $K \leq N_G(H)$)

2^{nd} isomorphism theorem: if $K, H \leq G$ s.t. $K \leq N_G(H)$,
such that

$$K/_{K\cap H} \cong HK/_{H}$$

Thm ("correspondence", "lattice", "3^{rd} iso" + "4^{th} iso")

if $H \trianglelefteq G$, there is a one-to-one correspondence

$$\{\text{subgroups } H \leq K \leq G\} \longleftrightarrow \{\text{subgroups } \overline{K} \leq \overline{G}\}$$

$\overline{G} = G/_H \xleftarrow{\pi} G$

$K \rightsquigarrow \overline{K} = \pi(K)$
$K = \pi^{-1}(\overline{K}) \leftsquigarrow \overline{K}$

(boils down to claim that 'if $f: G \to G'$ is a homomorphism, then $f(\text{subgroup}) = \text{subgroup}$, $f^{-1}(\text{subgroup}) = \text{subgroup}$)

Moreover: $K \leq K' \iff \overline{K} \leq \overline{K'}$ and $\exists$ bijection

K, K' are

$K'/_K \leftrightarrow \overline{K}'/_{\overline{K}}$

sandwiched between H and G

$$K'/K \cong \bar{K}'/\bar{K}$$

Moreover: $K \trianglelefteq K' \iff \bar{K} \trianglelefteq \bar{K}'$ and $\exists$ isomorphism

$$K'/K \cong \bar{K}'/\bar{K}$$

Proof of last moreover: $[g \bmod K] \longrightarrow [(g \bmod H) \bmod \bar{K}]$

$[g_1 \bmod K][g_2 \bmod K] = [g_1 g_2 \bmod K] \to [(g_1 g_2 \bmod H) \bmod \bar{K}] = [(g_1 \bmod H) \bmod \bar{K}][(g_2 \bmod H) \bmod \bar{K}]$

$K \trianglelefteq K' \iff \forall g \in K'$ we have $g K g^{-1} = K$ both sides are sandwiched between H and G

$\iff$ —— // —— $\pi(g K g^{-1}) = \pi(K)$

$\iff$ —— // —— $\pi(g) \cdot \bar{K} \cdot \pi(g)^{-1} = \bar{K}$

$\iff \forall \bar{g} \in \bar{K}'$ we have $\bar{g} \bar{K} \bar{g}^{-1} = \bar{K} \iff \bar{K} \trianglelefteq \bar{K}'$

Example: $G = \mathbb{Z}$, $H = n\mathbb{Z}$, $\bar{G} = \mathbb{Z}/n\mathbb{Z}$

$\{\text{subgroups } n\mathbb{Z} \leq K \leq \mathbb{Z}\} \longleftrightarrow \{\text{subgroups } \bar{K} \leq \mathbb{Z}/n\mathbb{Z}\}$

$\updownarrow K = m\mathbb{Z}$ $\qquad$ $\updownarrow \bar{K} = m\mathbb{Z}/n\mathbb{Z}$

$\{\text{divisors } m \mid n\} = \{\text{divisors } m \mid n\}$

Moreover: $K \leq K' \longleftrightarrow \bar{K} \leq \bar{K}'$

$K = m\mathbb{Z}$, $K' = m'\mathbb{Z}$ $\iff m' \mid m \mid n$

$\bar{K} = m\mathbb{Z}/n\mathbb{Z}$, $\bar{K}' = m'\mathbb{Z}/n\mathbb{Z}$ $\iff m' \mid m \mid n$

$m' \mid m \mid n \longleftrightarrow m' \mid m \mid n$